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THE FORCING EDGE-TO-VERTEX GEODETIC NUMBER OF A GRAPH

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Abstract: For a connected graph $G = (V, E)$, a set $S \subseteq E$ is called an *edge-to-vertex geodetic set* of G if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The minimum cardinality of an edge-to-vertex geodetic set of G is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G. A subset $T \subseteq S$ is called a *forcing* subset for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing edge-to-vertex geodetic number* of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing edge-to-vertex geodetic number* of G, denoted by $f_{ev}(G)$, is $f_{ev}(G) = min\{f_{ev}(S)\}$, where the minimum is taken over all minimum edge-to-vertex geodetic sets S in G. Some general properties satisfied by the concept forcing edge-to-vertex geodetic number is studied. The forcing edge-to-vertex geodetic number of certain classes of graphs are determined. It is shown that

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for every pair a, b of integers with $0 \leq a < b$, there exists a connected graph G such that $f_{ev}(G) = a$ and $g_{ev}(G) = b$.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic definitions and terminologies we refer to [1]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. Two vertices u and v of G are *antipodal* if $d(u, v) = diam\ G$ or $d(G)$. The *geodetic number* $g(G)$ of G is the minimum order of a geodetic set and any geodetic set of order $g(G)$ is called a *geodetic basis* of G. The geodetic number of a graph was introduced in [1] and further studied in [5]. For subsets A and B of $V(G)$, the distance $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d(A, B)$ is called an $A - B$ geodesic joining the sets $A, B \in V(G)$, where $u \in A$ and $v \in B$. A vertex x is said to *lie* on an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A - B$ geodesic as $uv - zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* of G if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The minimum cardinality of an edge-to-vertex geodetic set of G is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G or a $g_{ev}(G)$-set of G. The edge-to-vertex geodetic number of a graph was first introduced in [12] and further studied in [7,11]. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. An edge of a connected graph G is called an *extreme edge* of G if one of its end is an extreme vertex of G. For any edge e in a connected graph G, the *edge-to-edge eccentricity* $e_3(e)$ of e is $e_3(e) = max\ \{d(e, f) : f \in E(G)\}$. Any edge e for which $e_3(e)$ is minimum is called an *edge-to-edge central edge* of G and the set of all edge-to-edge central edges of G is the *edge-to-edge center* of G. The minimum eccentricity among the edges of G is the *edge-to-edge radius*, *rad* G and the maximum eccentricity among the edges of G is the *edge-to-edge diameter*, *diam* G of G. Two edges e and f are *antipodal* if $d(e, f) = diam\ G$ or $d(G)$. This concept was studied in [9]. The forcing concept was first introduced

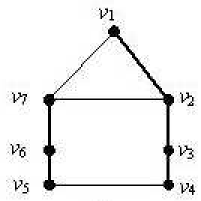


Figure 1.1

and studied in minimum dominating sets in [2]. And then the forcing concept is applied in various graph parameters viz. geodetic sets, hull sets, matching's, Steiner sets and edge covering in [3, 4, 6, 8, 10] by several authors. In this paper we study the forcing concept in minimum edge-to-vertex geodetic set of a connected graph.

Consider the graph G given in Figure 1.1 with $A = \{v_4, v_5\}$ and $B = \{v_1, v_2, v_7\}$, the paths $P : v_5, v_6, v_7$ and $Q : v_4, v_3, v_2$ are the only two $A - B$ geodesics so that $d(A, B) = 2$. For the graph G given in Figure 1.2, the three $v_1v_6 - v_3v_4$ geodesics are $P : v_1, v_2, v_3$; $Q : v_1, v_2, v_4$; and $R : v_6, v_5, v_4$ with each of length 2 so that $d\left(v_1v_6, v_3v_4\right) = 2$. Since the vertices v_2 and v_5 lie on the $v_1v_6 - v_3v_4$ geodesics P and R respectively, $S = \{v_1v_6, v_3v_4\}$ is an edge-to-vertex geodetic basis of G so that $g_{ev}\left(G\right) = 2$.

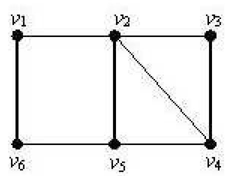


Figure 1.2

Throughout the following G denotes a connected graph with at least three vertices. The following Theorems are used in the sequel.

Theorem 1.1. (see [12]) *Let G be a connected graph with size q. Then every end-edge of G belongs to every edge-to-vertex geodetic set of G.*

Theorem 1.2. (see [12]) *For the complete bipartite graph $G = K_{m,n}(2 \leq$*

$m < n), g_{ev}(G) = n$.

Theorem 1.3. (see [12]) *If v is an extreme vertex of a connected graph G, then every edge-to-vertex geodetic set contains at least one extreme edge that is incident with v.*

2. The Forcing Edge-to-Vertex Geodetic Number of a Graph

Even though every connected graph contains a minimum edge-to-vertex geodetic set, some connected graph may contain several minimum edge-to-vertex geodetic sets. For each minimum edge-to-vertex geodetic set S in a connected graph G, there is always some subset T of S that uniquely determines S as the minimum edge-to-vertex geodetic set containing T. Such "forcing subsets" will be considered in this section.

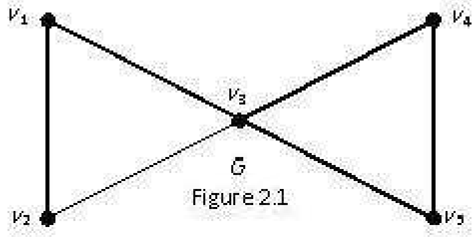
Definition 2.1. Let G be a connected graph and S an edge-to-vertex geodetic set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing edge-to-vertex geodetic number of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The forcing edge-to-vertex geodetic number of G, denoted by $f_{ev}(G)$, is $f_{ev}(G) = min\{f_{ev}(S)\}$, where the minimum is taken over all minimum edge-to-vertex geodetic sets S in G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1v_2, v_4v_5\}$ is the unique minimum edge-to-vertex geodetic set of G so that $f_{ev}(G) = 0$. For the graph G given in Figure 2.2, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev}-sets of G, such that $f_{ev}(S_1) = 2$, $f_{ev}(S_2) = f_{ev}(S_3) = 1$ so that $f_{ev}(G) = 1$.

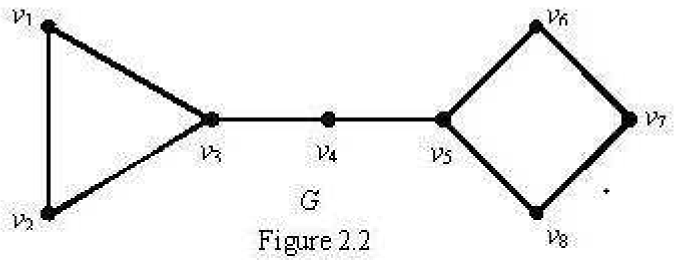
The next theorem follows immediately from the definition of the edge-to-vertex geodetic number and the forcing minimum edge-to-vertex geodetic number of a connected graph G.

Theorem 2.3. *For every connected graph G, $0 \leq f_{ev}(G) \leq g_{ev}(G)$.*

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2.1, $f_{ev}(G) = 0$ and for the graph $G = K_3$, $f_{ev}(G) = g_{ev}(G) = 2$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.2, $f_{ev}(G) = 1$ and $g_{ev}(G) = 3$ so that $0 < f_{ev}(G) < g_{ev}(G)$.



G
Figure 2.1



G
Figure 2.2

In the following, we characterize graphs G for which bounds in the Theorem 2.3 attained and also graph for which $f_{ev}(G) = 1$.

Theorem 2.5. *Let G be a connected graph. Then:*

a) $f_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex geodetic set.

b) $f_{ev}(G) = 1$ if and only if G has at least two minimum edge-to-vertex geodetic sets, one of which is a unique minimum edge-to-vertex geodetic set containing one of its elements, and

c) $f_{ev}(G) = g_{ev}(G)$ if and only if no minimum edge-to-vertex geodetic set of G is the unique minimum edge-to-vertex geodetic set containing any of its proper subsets.

The proof of the theorem is straight forward. So we can omitt the proof.

Definition 2.6. An edge e of a connected graph G is an edge-to-vertex geodetic edge of G if e belongs to every edge-to-vertex geodetic basis of G. If G has a unique edge-to-vertex geodetic basis S, then every edge of S is an edge-to-vertex geodetic edge of G.

Example 2.7. For the graph G given in Figure 2.1, $S = \{v_1v_2, v_4v_5\}$ is the unique minimum edge-to-vertex geodetic set of G so that both the edges in S are edge-to-vertex geodetic edges of G.

Remark 2.8. By Theorem 1.1, each end edge of G is an edge-to-vertex geodetic edge of G. In fact there are certain edge-to-vertex geodetic edges, which are not end edges of G as the following example shows.

Example 2.9. For the graph G given in Figure 2.2, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev}-sets of G so that every g_{ev}-set contains the edge v_1v_2. Hence the edge v_1v_2 is the unique edge-to-vertex geodetic edge of G, which is not an end edge of G.

Theorem 2.10. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum edge-to-vertex geodetic set of G. Then $\bigcap_{F \in \Im} F$ is the set of edge-to-vertex geodetic edges of G.*

Corollary 2.11. *Let G be a connected graph and S a minimum edge-to-vertex geodetic set of G. Then no edge-to-vertex geodetic edge of G belongs to any minimum forcing set of S.*

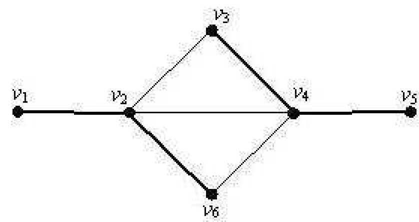
Theorem 2.12. *Let G be a connected graph and W be the set of all edge-to-vertex geodetic edges of G. Then $f_{ev}(G) \leq g_{ev}(G) - |W|$.*

Proof. Let S be a minimum edge-to-vertex geodetic set of G. Then $g_{ev}(G) = |S|$, $W \subseteq S$ and S is the unique minimum edge-to-vertex geodetic set containing $S - W$. Thus $f_{ev}(G) \leq |S - W| \leq |S| - |W| = g_{ev}(G) - |W|$. □

Corollary 2.13. *If G is a connected graph with k end edges, then $f_{ev}(G) \leq g_{ev}(G) - k$.*

Proof. This follows from Theorems 1.1 and 2.12. □

Remark 2.14. The bound in Theorem 2.12 is sharp. For the graph G given in Figure 2.3, $S_1 = \{v_1v_2, v_2v_3, v_4v_5, v_4v_6\}$, $S_2 = \{v_1v_2, v_3v_4, v_4v_5, v_4v_6\}$, $S_3 = \{v_1v_2, v_2v_3, v_4v_5, v_2v_6\}$ and $S_4 = \{v_1v_2, v_3v_4, v_4v_5, v_2v_6\}$ are the only four minimum edge-to-vertex geodetic sets of G such that $f_{ev}(S_1) = f_{ev}(S_2) = f_{ev}(S_3) = f_{ev}(S_4) = 2$ so that $f_{ev}(G) = 2$ and $g_{ev}(G) = 4$. Also, $W = \{v_1v_2, v_4v_5\}$ is the set of all edge-to-vertex geodetic edges of G and so $f_{ev}(G) = g_{ev}(G) - |W|$.



G

Figure 2.3

Also, the inequality in Theorem 2.12 can be strict. For the graph G given in Figure 2.2, $g_{ev}(G) = 3$ and $f_{ev}(S_2) = f_{ev}(S_3) = 1$ and $f_{ev}(S_1) = 2$ so that $f_{ev}(G) = 1$. Here, v_1v_2 is the only edge-to-vertex geodetic edge of G and so $f_{ev}(G) < g_{ev}(G) - |W|$.

In the following we determine the forcing edge-to-vertex geodetic number of some standard graphs.

Theorem 2.15. *For an even cycle $C_p(p \geq 4)$, a set $S \subseteq E(G)$ is a minimum edge-to-vertex geodetic set if and only if S consists of antipodal edges.*

Proof. Let $p = 2k$ and let $C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1$ be the cycle. Then the edges v_1v_2 and $v_{k+1}v_{k+2}$ are antipodal edges. Let $S = \{v_1v_2, v_{k+1}v_{k+2}\}$. Clearly, S is a minimum edge-to-vertex geodetic set of C_p. Conversely, let S be a minimum edge-to-vertex geodetic set of C_p. Then $g_{ev}(C_p) = |S|$. Let $S^{'}$ be any set of pair of antipodal edges of C_p. Then as in the first part of this theorem, $S^{'}$ is a minimum edge-to-vertex geodetic set of C_p. Hence $|S^{'}| = |S|$. Thus $S = \{uv, xy\}$. If uv and xy are not antipodal, then any vertex that is not on the $uv - xy$ geodesic does not lie on the $uv - xy$ geodesic. Thus S is not a minimum edge-to-vertex geodetic set, which is a contradiction. □

Theorem 2.16. *For the cycle $C_p(p \geq 4)$,* $f_{ev}(C_p) = \left\{ \begin{array}{c} 1 \;\; if\ p\ is\ even \\ 2 \;\; if\ p\ is\ odd \end{array} \right\}$.

Proof. If p is even, then by Theorem 2.15, every minimum edge-to-vertex geodetic set of C_p consists of pair of antipodal edges. Hence C_p has $p/2$ independent minimum edge-to-vertex geodetic sets and it is clear that each singleton set is the minimum forcing set for exactly one minimum edge-to-vertex geodetic set of C_p. Hence it follows from Theorem 2.5 (a) and (b) that $f_{ev}(C_p) = 1$.

Let p be odd. Let $p = 2n+1$. Let the cycle be $C_p : v_1, v_2, v_3, ..., v_{2n+1}, v_1$. If $S = \{uv, xy\}$ is any set of two edges of C_p, then no edge of the $uv - xy$ longest path lies on the $uv - xy$ geodesic in C_p and so no two element subset of C_p is an edge-to-vertex geodetic set of C_p. Now, it clear that the sets

$$S_1 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n}v_{2n+1}\},$$

$$S_2 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n+1}v_1\},$$

$$S_3 = \{v_2v_3, v_{n+2}v_{n+3}, v_{2n+1}v_1\}, ...,$$

$$S_{2n} = \{v_nv_{n+1}, v_{2n}v_{2n+1}, v_{n-1}v_n\},$$

$$S_{2n+1} = \{v_{n+1}v_{n+2}, v_{2n+1}v_1, v_{n-1}v_n\}$$

are the minimum edge-to-vertex geodetic sets of C_p. (Note that there are more minimum edge-to-vertex geodetic sets of C_p, for example

$$S = \{v_{n+2}v_{n+3}, v_1v_2, v_nv_{n+1}\}$$

is a minimum edge-to-vertex geodetic set different from these). It is clear from the minimum edge-to-vertex geodetic sets $S_i\,(1 \le i \le 2n+1)$ that each $\{v_iv_{i+1}\}\,(1 \le i \le 2n)$ and $\{v_{2n+1}v_1\}$ is a subset of more than one minimum edge-to-vertex geodetic set $S_i(1 \le i \le 2n+1)$. Hence it follows from Theorem 2.5 (a) and (b) that $f_{ev}(C_p) \ge 2$. Since S_1 is the unique minimum edge-to-vertex geodetic set containing $T = \{v_{n+1}v_{n+2}, v_{2n}v_{2n+1}\}$, it follows that $f_{ev}(S_1) = 2$. Thus $f_{ev}(C_p) = 2$. □

Theorem 2.17. *For the complete graph $G = K_p(p \ge 4)$ with p even, a set S of edges of G is a minimum edge-to-vertex geodetic set of G if and only if S consists of $p/2$ independent edges.*

Proof. Let S be any set of $p/2$ independent edges of K_p. Since each vertex of K_p is incident with an edge of S, it follows that $g_{ev}(G) \le p/2$. If $g_{ev}(G) < p/2$, then there exists a minimum edge-to-vertex geodetic set S' of K_p such that $|S'| < p/2$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of S'. Hence v is neither incident with any edge of S' nor lies on a geodesic joining a pair of edges of S' and so S' is not a minimum edge-to-vertex geodetic set of G, which is a contradiction. Thus S is a minimum edge-to-vertex geodetic set of K_p.

Conversely, let S be a minimum edge-to-vertex geodetic set of K_p. Let S' be any set of $p/2$ independent edges of K_p. Then as in the first part of this theorem, S' is a minimum edge-to-vertex geodetic basis of K_p. Therefore $\left|S'\right| = p/2$. Hence $|S| = p/2$. If S is not independent, then there exists a vertex v of K_p such that v is not incident with any edge of S and it follows that S

is not a minimum edge-to-vertex geodetic set of G, which is a contradiction. Therefore, S consists of $p/2$ independent edges. □

Theorem 2.18. *For the complete graph $G = K_p(p \geq 4)$ with p even, $f_{ev}(G) = \frac{P-2}{2}$.*

Proof. Let S be a minimum edge-to-vertex geodetic set of G such that $|S| = p/2$. Then by Theorem 2.17, every element of S is independent. We show that $f_{ev}(G) = \frac{P}{2} - 1$. Suppose that $f_{ev}(G) \leq \frac{P}{2} - 2$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| \leq \frac{P}{2} - 2$. Hence there exists at least two edges $u_iu_j,\ u_lu_m \in S$ such that $u_iu_j, u_lu_m \notin T$ and $i \neq l, j \neq m$. Then $S_1 = S - \{u_iu_j, u_lu_m\} \cup \{u_iu_m, u_lu_j\}$ is a set of $p/2$ independent edges of G containing T. By Theorem 2.16, S_1 is a minimum edge-to-vertex geodetic set of G which is a contradiction to T is a forcing subset of S. Hence $f_{ev}(G) = \frac{P}{2} - 1 = \frac{P-2}{2}$. □

Theorem 2.19. *For the complete graph $G = K_p(p \geq 5)$ with p odd, a set S of edges of G is a minimum edge-to-vertex geodetic set of G if and only if S consists of $\frac{P-3}{2}$ independent edges and two adjacent edges of G.*

Proof. Let S_1 be any set of $\frac{P-3}{2}$ independent edges of K_p and S_2 be two adjacent edges of K_p, each of which is independent with the edges of S_1. Let $S = S_1 \cup S_2$. Since each vertex of K_p is incident with an element of S, it follows that S is a minimum edge-to-vertex geodetic set of K_p so that $g_{ev}(G) \leq \frac{P-3}{2} + 2 = \frac{P+1}{2}$. If $g_{ev}(G) < \frac{P+1}{2}$, then there exists a minimum edge-to-vertex geodetic set S' of K_p such that $|S'| < \frac{P+1}{2}$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of S'. Hence the vertex v is neither incident with any edge of S' nor lies on a geodesic joining a pair of edges of S' and so S' is not a minimum edge-to-vertex geodetic set of G, which is a contradiction. Hence $g_{ev}(G) = \frac{P+1}{2}$.

Conversely, let S be a minimum edge-to-vertex geodetic set of G. Let S' be any set of $\frac{P-3}{2}$ independent edges of G and two adjacent edges of G. Then as in the first part of this theorem, S' is a minimum edge-to-vertex geodetic set of G. Therefore, $|S'| = \frac{P+1}{2}$. Hence $|S| = \frac{P+1}{2}$. Let us assume that $S = S_1 \cup S_2$, where S_1 consists of independent edges and S_2 consists of adjacent edges of G. If $|S1| \leq \frac{P-3}{2} - 1$, then S_2 must contain at most $n - |S_1|$ edges. Then there exists at least one vertex v of K_p such that v is not incident with any edge of S and so S is not a minimum edge-to-vertex geodetic set of G, which is a contradiction. Therefore S consists of $\frac{P-3}{2}$ independent edges of G and two adjacent edges of G. □

Theorem 2.20. *For the complete graph $G = K_p (p \geq 5)$ with p odd, $f_{ev}(G) = \frac{P-1}{2}$.*

Proof. Let S be a minimum edge-to-vertex geodetic set of G. Then by Theorem 2.19, $S = S_1 \cup S_2$, where S_1 consists of $\frac{P-3}{2}$ independent edges and S_2 consists of two adjacent edges and $|S| = \frac{P+1}{2}$. We show that $f_{ev}(G) = \frac{P+1}{2} - 1$. Suppose that $f_{ev}(G) \leq \frac{P+1}{2} - 2$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| \leq \frac{P+1}{2} - 2$. Hence there exists at least two edges $x, y \in S$ such that $x, y \notin T$. Let us assume that $S_2 = \{u_x u_y, u_y u_z\}$. Suppose that $x, y \in S_1$. Then $x = u_i u_j$ and $y = u_l u_m$ such that $i \neq l, j \neq m$. Now, $S_3 = S - \{x, y\} \cup \{u_i u_m, u_l u_j\}$ consists of $\frac{P-3}{2}$ independent edges and two adjacent edges of G containing T. By Theorem 2.19, S_3 is a minimum edge-to-vertex geodetic set of G containing T, which is a contradiction to T is a forcing subset of G. Suppose that $x, y \in S_2$. Let $x = u_x u_y$ and $y = u_y u_z$. Let $u_i u_j$ be an edge of S_1. Now, join the vertices u_y, u_i and u_z, u_j. Now $S_4 = S_1 - \{u_i u_j\} \cup \{u_x u_y\} \cup \{u_y u_i, u_z u_j\}$ consists of $\frac{P-3}{2}$ independent edges and two adjacent edges of G. By Theorem 2.19, S_4 is a minimum edge-to-vertex geodetic set of G containing T, which is a contradiction. Suppose that $x \in S_1$ and $y \in S_2$. Let $x = u_i u_j$ and $y = u_x u_y$. Now, $S_5 = S_1 - \{u_i u_j\} \cup \{u_j u_y\} \cup \{u_i u_x, u_y u_z\}$ consists of $\frac{P-3}{2}$ independent edges and two adjacent edges of G containing T. By Theorem 2.19, S_5 is a minimum edge-to-vertex geodetic set of G, which is a contradiction to that T is a forcing subset of G. Hence $f_{ev}(G) = \frac{P+1}{2} - 1 = \frac{P-1}{2}$. □

Theorem 2.21. *A set S of edges of $G = K_{n,n} (n \geq 2)$ is a minimum edge-to-vertex geodetic set of G if and only if S consists of n independent edges.*

Proof. The proof is similar to the proof of Theorem 2.17. □

Theorem 2.22. *For the complete bipartite graph $G = K_{n,n} (n \geq 2)$, $f_{ev}(G) = n - 1$.*

Proof. The proof is similar to the proof of Theorem 2.18. □

Theorem 2.23. A set S of edges of $G = K_{m,n}(2 \leq m < n)$ a minimum edge-to-vertex geodetic set of G if and only if S consists of $m-1$ independent edges of G and $n-m+1$ adjacent edges of G.

Proof. The proof is similar to the proof of Theorem 2.19. □

Theorem 2.24. *For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, $f_{ev}(G) = n-1$.*

Proof. The proof is similar to the proof of Theorem 2.20. □

Theorem 2.25. *For a non trivial tree of size $q \geq 2$, $f_{ev}(G) = 0$.*

Proof. For $G = K_{1,q}$, it follows from Theorem 1.1 that the set of all end edges of G is the unique minimum edge-to-vertex geodetic set of G. Now, it follows from Theorem 2.5(a) that $f_{ev}(G) = 0$. □

3. Realization Result

In view of Theorem 2.3, we have the following realization theorem.

Theorem 3.1. *For every pair a, b of integers with $0 \leq a < b$, there exists a connected graph G such that $f_{ev}(G) = a$ and $g_{ev}(G) = b$.*

Proof. Suppose $a = 0$. Let $G = K_{1,b}$. Then by Theorem 2.25, $f_{ev}(G) = 0$ and from Theorem 1.1, $g_{ev}(G) = b$. Suppose that $b = a+1$. Let $G = K_{2,b}$. Then by Theorem 1.2, $g_{ev}(G) = b$ and from Theorem 2.24, $f_{ev}(G) = b-1 = a$. Thus, we assume that $0 < a < b$. Let $F_i : u_i, v_i, x_i, u_i (1 \leq i \leq a)$ be a copy of C_3. Let G be the graph obtained from $F_i (1 \leq i \leq a)$ by first identifying the vertices x_{i-1} of F_{i-1} and x_i of $F_i (2 \leq i \leq a)$ and then adding $b-a$ new vertices $z_1, z_2, ..., z_{b-a-1}, u$ and joining the $b-a$ edges $u_1 z_i (1 \leq i \leq b-a-1)$ and $x_a u$. The graph G is given in Figure 3.1. Let $Z = \{u_1 z_1, u_1 z_2, ..., u_1 z_{b-a-1}, x_a u\}$ be the set of all end edges of G. Let $Hi = \{h_i, k_i\}$ $(1 \leq i \leq a)$, where $h_i = u_i v_i$ and $k_i = v_i x_i$. First we show that $g_{ev}(G) = b$. By Theorem 1.3, every edge- to-vertex geodetic set of G must contain at least one vertex from $H_i (1 \leq i \leq a)$. Thus $g_{ev}(G) \geq b-a+a = b$. On the other hand, since the set $S = Z \cup \{h_1, h_2, ..., h_a\}$ is a minimum edge-to-vertex geodetic set of G, it follows that $g_{ev}(G) \leq |S| = b$.

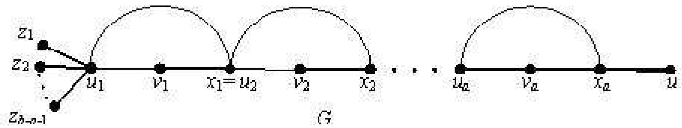


Figure 3.1

Thus $g_{ev}(G) = b$. Next we show that $f_{ev}(G) = a$. Since every g_{ev}-set of G contains Z, it follows from Theorem 2.12 that $f_{ev}(G) \leq g_{ev}(G) - |Z| = b - (b - a) = a$. Now, since $g_{ev}(G) = b$ and every minimum edge-to-vertex geodetic set of G contains S, it is easily seen that every minimum edge-to-vertex geodetic set W is of the form $W \cup \{e_1, e_2, ..., e_a\}$, where $e_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exists an edge $e_j (1 \leq j \leq a)$ such that $ej \notin T$. Let f_j be an edge of H_j distinct from e_j. Then $W_1 = (S - \{e_j\}) \cup \{f_j\}$ is a g_{ev}-set properly containing T. Thus W is not the unique g_{ev}-set containing T. Thus T is not a forcing subset of S. This is true for all minimum edge-to-vertex geodetic sets of G and so it follows that $f_{ev}(G) = a$. □

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